Mathematical Review:
Vector, Electric Field and Electric Potential

1 The Vectors

In these notes, we will use the basis $\hat{i}, \hat{j}, \hat{k}$ to denote the *unit vectors* pointing in the x, y, z directions. The term *unit vector* means the magnitude of the vector is 1.

We will use the two vectors, $\vec{u}, \vec{v}$ in most of our examples below. They are defined as:

$$\vec{u} = a\hat{i} + b\hat{j} + c\hat{k}$$
$$\vec{v} = d\hat{i} + e\hat{j} + f\hat{k}$$

2 Magnitude

The magnitude of a vector $\vec{u}$ is denoted as $|\vec{u}|$ or $|u|$ or simply u. It can be computed simply as follows:

$$|\vec{u}| = \sqrt{a^2 + b^2 + c^2}$$

3 Dot product

The dot product is defined by the following relation:

$$\Rightarrow \vec{u} \cdot \vec{v} = ad + be + cf = |\vec{u}||\vec{v}|cos\theta$$

where θ is the angle between the two vectors.

Note that the dot product combines two *vectors* to form a *scalar*, and that $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$.

Question 3.1 What is the difference between a vector and a scalar?

Question 3.2. Find the angle between the two vectors $\vec{u} = 3\hat{i} + 5\hat{j} + 9\hat{k}$ and $\vec{v} = 2\hat{i} - 1\hat{j} + 1\hat{k}$.

Question 3.3.

(a) Show that $\vec{u} \cdot \vec{u} = |\vec{u}|^2$. Prove this for all $\vec{u}$, not just for the one in *question 3.2.* Do not assume $\vec{u} = 3\hat{i} + 5\hat{j} + 9\hat{k}$.

(b) Show that $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = |\vec{u}|^2 + 2\vec{u} \cdot \vec{u} + |\vec{v}|^2$.

Question 3.4 If $\vec{u} \cdot \vec{v} = 0$, what does that tell you about the vectors $\vec{u}$ and $\vec{v}$? Show proof.

A lot of relations in physics can be expressed in dot products (here we assume constant $\vec{F}$ and $\vec{E}$):

$$\begin{aligned} \text{work done} \quad W &= \vec{F} \cdot \vec{s} \\ \text{power} \quad P &= \vec{F} \cdot \vec{v} \\ \text{electric flux} \quad \Phi &= \vec{E} \cdot \vec{A} \\ \text{potential difference} \quad \Delta V &= -\vec{E} \cdot \vec{s} \end{aligned}$$

Question 3.5. Find the work done of a force $\vec{F} = (1.2\hat{\imath} - 4.3\hat{\jmath} - 2.8\hat{k})N$ under a displacement of $\vec{s} = (6\hat{\imath} + 2\hat{\jmath} + 1\hat{k})m$. Do not forget the units.

Question 3.6. Suppose $\vec{w} = \vec{u} - \vec{v}$, and you are told θ is the angle between $\vec{u}$ and $\vec{v}$, write $|\vec{w}|$ in terms of θ, $|\vec{u}| = u$ and $|\vec{v}| = v$. [*Hint: You should first compute $\vec{w} \cdot \vec{w}$ and then use the result of 3.3. Do not write $\vec{u}$ and $\vec{v}$ in components!*]

Question 3.7. If $\vec{u}, \vec{v}$ represents two sides of a triangle, show that the third side can be represented by the vector $\vec{w} = \vec{u} - \vec{v}$.

The results of *3.6* and *3.7* gives you a formula to compute the length of the third side of a triangle given the other two sides and the angle between them. It is important to note that this formula works even if the triangle is not a right angle triangle.

Question 3.8. What is the answer of *3.6* if $\theta = 90°$? What if $\theta = 0°$?

Question 3.9. Given a triangle with two sides measuring $6m$ and $9m$, and the angle between them $25°$, find the length of the third side.

Question 3.10. If $\hat{w}$ is a unit vector, show that $\hat{w} \cdot \hat{w} = 1$.

4 The Gradient Operator $\vec{\nabla}$

The operator $\vec{\nabla}$, which is sometimes written as *grad*, changes a function into a *vector*. Given any function $f(x, y, z)$, $\vec{\nabla} f$ is defined as:

$$\vec{\nabla} f \equiv \partial_x f \hat{\imath} + \partial_y f \hat{\jmath} + \partial_z f \hat{k}$$

where $\partial_x = \frac{\partial}{\partial x}$, ..., etc. are the partial derivatives.

Electric potential V (a scalar) is related to the electric field $\vec{E}$ (a vector) via the relation:

$$\vec{E} = -\vec{\nabla} V$$

Question 4.1. Given $V = 12\frac{xy^2}{z^4}$, find the electric field $\vec{E}$. What is the magnitude of the electric field at coordinate $(3, 0.5, 1)$?

Question 4.2. This question assumes only two dimensions (instead of three). Given $V = -\frac{1}{2}(x^2 + y^2)$, find the electric field $\vec{E}$. Draw $\vec{E}$ at the following points:
$(1, 0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, 1), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-1, 0), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (0, -1)$.

Suppose you are interested in the function $f(x, y, z)$ at the point $(x + \delta x, y + \delta y, z + \delta z)$. We have the following relation:

$$\delta f \equiv f(x + \delta x, y + \delta y, z + \delta z) - f(x, y, z) = \delta x \partial_x f + \delta y \partial_y f + \delta z \partial_z f$$

Question 4.3. Define $\vec{\delta r} = \delta x\hat{i} + \delta y\hat{j} + \delta z\hat{k}$, show that $\delta f = \vec{\delta r} \cdot \vec{\nabla} f$.

Question 4.4. Show that $\delta V = -\vec{\delta r} \cdot \vec{E}$. In other words, you can calculate the change in the potential once you know $\vec{E}$.

5 The Line Integral

A line integral of a vector $\vec{u}$ from point A to B is written as:

$$I = \int_A^B \vec{u} \cdot d\vec{r}$$

In general, this integral depends on the path you take from A to B. However, if the vector you are integrating can be written as the *gradient* (or $\vec{\nabla}$) of something then the path will no longer matter. One good example is the electric field $\vec{E} = -\vec{\nabla} V$, so when you integrate $\vec{E}$, it does not matter what paths you take.

Earlier we showed that $\delta f = \vec{\nabla} f \cdot \vec{\delta r}$. Putting $f = V$ and integrating both sides gives:

$$\begin{aligned} \Delta V_{BA} \equiv V_B - V_A &= \int_A^B \vec{\nabla} V \cdot d\vec{r} \\ \Rightarrow \Delta V_{BA} &= -\int_A^B \vec{E} \cdot d\vec{r} \end{aligned}$$

Since the path of the integral does not matter, it is usually easiest to pick a straight line. A line integral along a straight line is relatively easy to perform.

Let us first look at the special case of constant $\vec{E}$. In this case:

$$\Delta V_{BA} = -\int_A^B \vec{E} \cdot d\vec{r} = -\vec{E} \cdot \int_A^B d\vec{r} \Rightarrow \Delta V_{BA} = -\vec{E} \cdot \vec{r}_{BA}$$

where $\vec{r}_{BA} = \vec{r}_B - \vec{r}_A$.

Question 4.5. Find the potential difference ΔV_{BA} between point A at $(1, 0, 0)$ and point B at $(2, 3, 1)$ for a uniform field of $\vec{E} = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$.

Next we will look at the case of point charge, which has spherical symmetry. We know that $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$, where $\hat{r}$ is the unit radial vector. We want to look at ΔV_{BA} where point A is at infinity, and point B is distance r away from the charge.

Question 4.6. What is $\vec{E} \cdot \hat{r}$?

The way to do the line integral is to replace $d\vec{r}$ in the integral by $\hat{r}dr$, so we have:

$$\Delta V_{BA} = -\int_{\infty}^{r} \vec{E} \cdot \hat{r} dr$$

Question 4.7. Perform the r integral above to find ΔV_{BA}.

Remember that $\Delta V_{BA} \equiv V_B - V_A$. Since A is at infinity, the convention is to put $V_A = V_\infty = 0$. Therefore we have $\Delta V_{BA} \equiv V_B$.

Question 4.8. Write down the potential ($V_B = V(r)$) at distance r from a charge q.