Mathematical Review for AC Circuits: Complex Number

1 Notation

When a number x is real, we write $x \in \mathbb{R}$.
When a number z is complex, we write $z \in \mathbb{C}$.
Complex conjugate of z is written as z^* here. Some books use the notation $\overline{z}$.
The modulus of z is written as $|z|$.

2 Complex Number

A complex number $z \in \mathbb{C}$ can be written in general as:

$$z = a + ib = Re^{i\theta}$$

where $i = \sqrt{-1}$, and $a, b, R, \theta \in \mathbb{R}$.

In particular, we have:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1}$$

Note that the variable θ above is a dummy variable, meaning that eqn(1) remains true for any θ, for example, $e^{ianything} = \cos anything + i \sin anything$, $e^{7i\theta} = \cos 7\theta + i\sin 7\theta$, $e^{ix^3} = \cos x^3 + i\sin x^3$, and $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$.

1. If $z = a + ib = Re^{i\theta}$, write a, b in terms of R, θ. Also write R, θ in terms of a, b.

   Note that a is called the real part of z and b the imaginary part, and we often use the notation:

   $$a = \Re(z) \quad b = \Im(z)$$

2. Write $e^{-i\theta}$ in terms of $\cos\theta$ and $\sin\theta$. Combine this result with eqn(1) to show:

   $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

3. Use the same approach as the previous question to find a similar expression for $\sin\theta$ and $\tan\theta$ in terms of $e^{\pm i\theta}$.

4. Use the results from the two previous questions to prove the following identities:

$$\begin{aligned} \cos^2\theta + \sin^2\theta &= 1 \\ 2\cos^2\theta - 1 = 1 - 2\sin^2\theta &= \cos(2\theta) \\ 2\sin\theta\cos\theta &= \sin(2\theta) \end{aligned}$$

5. By comparing the real and imaginary parts of $e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$, show:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

6. What are $e^{i\pi}$ and $e^{i\pi/2}$?
7. Simplify $i^2, i^3, i^4, i^5, i^{19}, i^{120}, \frac{1}{i}$.
8. What are the real part and imaginary part of $ie^{i\theta}$?
9. Show that $ie^{i\theta} = e^{i(\theta+\pi/2)}$ using the result of Question 6. By using eqn(1) and comparing the real part of the left and the real part of the right, show that $\cos(\theta + \pi/2) = -\sin\theta$. Find also $\sin(\theta + \pi/2)$ by comparing the imaginary part.
10. (Optional) Evaluate $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos N\theta$.

3 The Complex Plane

A complex number $z = a + ib$ can be represented on the complex plane by an arrow pointing from the origin to the point (a, b).

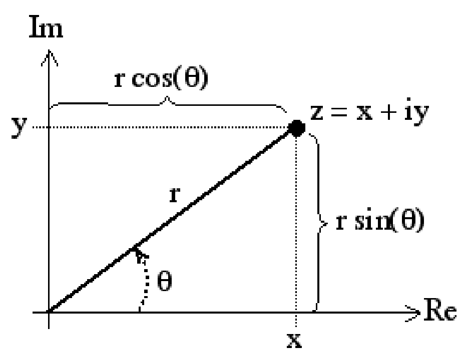

Figure 1

11. 3.1 Represent the following complex numbers on a complex plane:
$2 + 4i, -1 - 2i, 4 - i, e^{i\pi}, 2e^{i\pi/6}, e^{-i\pi/3}, e^{3i\pi}$

4 The Modulus and the Complex Conjugate

The modulus (or the absolute value, magnitude) of a complex number $z = a + ib$ is give by:

$$|z| = \sqrt{a^2 + b^2}$$

The complex conjugate of a complex number $z = a + ib$ is give by:

$$z^* = a - ib$$

In general, the complex conjugate is found by simply replacing every i with $-i$. For example, if $z = e^{i\theta}$, then $z^* = e^{-i\theta}$. If $z = \frac{1}{1-7ie^{17i}}$ then $z^* = \frac{1}{1+7ie^{-17i}}$. We also have $(z_1 z_2)^* = z_1^* z_2^*$.

One rule that is much more useful than it seems is the following:

$$|z_1 z_2| = |z_1||z_2|$$

In particular, if $|w| = 1$ (i.e., any complex numbers of the form $e^{i\theta}$), we have:

$$|z_1 w| = |z_1||w| = |z_1| \tag{2}$$

12. Show that $|z|^2 = z^* z$.
13. Find the complex conjugate and the modulus of $Re^{i\theta}$.
14. Represent the complex conjugate of the following complex numbers on a complex plane:
    $2 + 4i, 4 - i, e^{i\pi}, 2e^{-i\pi/3}$
15. Find the magnitude of $z = 1 + e^{i\phi}$. Hint: Do not use $|z| = \sqrt{a^2 + b^2}$, instead you should use $|z|^2 = z^* z$. Your answer should contain a cosine function.
16. Find the magnitude of $z = e^{i\phi/2} + e^{-i\phi/2}$. You should use the result of Question 2 for cosine to solve this problem (the answer involves $\cos\frac{\phi}{2}$). When you are done, argue with the help of eqn(2) why the magnitude of $1 + e^{i\phi}$ and $e^{i\phi/2} + e^{-i\phi/2}$ are identical. Combine your answer here with that of Question 15 to prove $2\cos^2\theta - 1 = \cos(2\theta)$.
17. Given two complex numbers $u = re^{i(\alpha+\theta)}$ and $v = se^{i\alpha}$, find the magnitude of $w = u - v$ in terms of r, s, θ.
18. Represent the u, v, w on the complex plane. What is the geometrical meaning of your result from the previous question? Hint: It has to do with the lengths of a triangle.

5 The Calculus of Complex Numbers

19. Find $\frac{d}{d\theta}z$ for $z = e^{i\theta}$. What is the real part and the imaginary parts of $\frac{d}{d\theta}z$? Are they the same as $\frac{d}{d\theta}\Re(z)$ and $\frac{d}{d\theta}\Im(z)$? In other words, I want you to prove that $\Re(\frac{d}{d\theta}z) = \frac{d}{d\theta}\Re(z)$ and $\Im(\frac{d}{d\theta}z) = \frac{d}{d\theta}\Im(z)$.

20. Prove $\Re(\frac{d^2}{d^2\theta}z) = \frac{d^2}{d^2\theta}\Re(z)$ and $\Im(\frac{d^2}{d^2\theta}z) = \frac{d^2}{d^2\theta}\Im(z)$ for $z = e^{i\theta}$.

21. Find $\int d\theta z$ for $z = e^{i\theta}$. Prove $\Re(\int d\theta z) = \int d\theta \Re(z)$ and $\Im(\int d\theta z) = \int d\theta \Im(z)$.

22. Find $\frac{d}{dt}z$ for $z = e^{i\omega t}$. Prove that $\Re(\frac{d}{dt}z) = \frac{d}{dt}\Re(z)$ and $\Im(\frac{d}{dt}z) = \frac{d}{dt}\Im(z)$.

23. Find $\int dt z$ for $z = e^{i\omega t}$. Prove $\Re(\int dt z) = \int dt \Re(z)$ and $\Im(\int dt z) = \int dt \Im(z)$.

6 AC Circuit

The current in an ac circuit is often given by an equation $I = I_0 \sin(\omega t + \phi)$, where ω is the angular frequency and ϕ is the phase. It turns out it is often convenient to use complex numbers to represent this as:

$$I = I_0 e^{i(\omega t + \phi)} = (I_0 e^{i\phi}) e^{i\omega t}$$

In other words, every time you see and alternating current, simply do the replacement $I = I_0 \sin(\omega t + \phi) \to I = I_0 e^{i(\omega t + \phi)}$. The same goes for voltage V and charge q, for example: $V = V_0 \sin(\omega t + \phi) \to V = V_0 e^{i(\omega t + \phi)}$.

Likewise, whenever someone says the current is $I_0 e^{i(\omega t + \phi)}$, you should know that what he *really* means is that the current is $I = \Im(I_0 e^{i(\omega t + \phi)}) = I_0 \sin(\omega t + \phi)$. Using complex number to denote alternating current, voltage and charge is very common and is very powerful mathematically.

Just as the time average of $\sin \omega t$ and $\cos \omega t$ are zero. We have:

$$\langle e^{i\omega t} \rangle = 0$$

where $\langle e^{i\omega t} \rangle$ denotes the time average of the complex number.

24. What is the modulus of $I = I_0 e^{i(\omega t + \phi)}$?

25. Recall $V = IR$. Suppose $I = I_0 e^{i\omega t}$, find V in terms of complex number. What is the imaginary part of V?

26. Recall $I = \frac{dq}{dt} \Rightarrow q = \int dt I$. Suppose $I = I_0 e^{i\omega t}$, find q in terms of complex number. What is the imaginary part of q?

27. Recall $V_C = \frac{q}{C}$ for a capacitor. Suppose $I = I_0 e^{i\omega t}$, find V_C in terms of complex number. Define the *complex* reactance as $X_C = \frac{V_C}{I}$ (all variables here are complex numbers) find X_C. What is the imaginary part of V_C? You can use your answer to 26.

28. Recall $V_L = L\frac{dI}{dt}$ for an inductor. Suppose $I = I_0 e^{i\omega t}$, find V_L in terms of complex number. Define the *complex* reactance as $X_L = \frac{V_L}{I}$ (all variables here are complex numbers) find X_L. What is the imaginary part of V_L?

29. If $V = IR + \frac{q}{C} + L\frac{dI}{dt}$, suppose $I = I_0 e^{i\omega t}$, find V in terms of complex number. Define the total *complex* reactance as $X = \frac{V}{I}$ (all variables here are complex numbers) find X. Find also the modulus of X.